Conserved quantities in the noncommutative principal chiral model with Wess–Zumino term

2005 J. Phys. A: Math. Gen. 38 9241

(http://iopscience.iop.org/0305-4470/38/42/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 9241–9250 doi:10.1088/0305-4470/38/42/005

Conserved quantities in the noncommutative principal chiral model with Wess–Zumino term

U Saleem, M Hassan and M Siddiq[1]

Department of Physics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan

E-mail: usman-physics@yahoo.com, mhassan@physics.pu.edu.pk and mohsin-pu@yahoo.com

Received 9 May 2005, in final form 7 September 2005
Published 5 October 2005
Online at stacks.iop.org/JPhysA/38/9241

Abstract
We construct a noncommutative extension of the $U(N)$ principal chiral model with Wess–Zumino term and obtain an infinite set of local and non-local conserved quantities for the model using the iterative procedure of Brezin *et al* (1979 *Phys. Lett.* B **82** 442). We also present the equivalent description as a Lax formalism of the model. We expand the fields perturbatively and derive zeroth- and first-order equations of motion, zero-curvature condition, iteration method, Lax formalism, local and non-local conserved quantities.

PACS numbers: 11.10.Nx, 02.30.Ik

During the last 20 years, the subject of classical and quantum integrability of field theoretic models has attracted a great interest [1–17]. The integrability of a field theoretic model is guaranteed by the existence of an infinite number of local and non-local conserved quantities [2–17]. These conserved quantities appear due to the fact that the field equations of these models can be expressed as a zero-curvature condition for a given connection.

There is an increasing interest in noncommutative geometry [18] due to its applications in particle physics, string theory, differential geometry etc [19–24]. Over the last few years, there has been an increasing interest in a noncommutative[2] extension of the field theories [25–38]. The noncommutative version of the integrable field theoretic models is obtained

[1] On study leave from PRD (PINSTECH) Islamabad, Pakistan.

[2] The noncommutative spaces are defined by the noncommutativity of the coordinates, i.e.

$$[x^i, x^j] = \mathrm{i}\theta^{ij},$$

where θ^{ij} known as deformation parameters are real constants.

by replacing the ordinary product of fields in the action by their $\star$-product [39][3]. This has shown that in general the noncommutativity of time variables leads to non-unitarity and affects the causality of the theory [23, 24]. These noncommutative integrable models reduce to ordinary models in the limit when the deformation parameter θ vanishes. The noncommutative extension of integrable models such as the principal chiral model with and without a Wess–Zumino term, Liouville equation, sine-Gordon equation, Korteweg–de Vries (KdV) equation, Boussinesq equation, Kadomtsev Petviashvili (KP) equation, Sawada–Kotera equation, nonlinear Schrödinger equation and Burgers equation have been studied by different authors by exploiting different techniques [25–37]. The noncommutative sigma model, in particular the $U(N)$ principal chiral model, has been investigated in [26, 27] and some results regarding their integrability have been obtained. The ultraviolet property of the Wess–Zumino–Witten (WZW) model has been studied in [25].

In this paper we present a noncommutative extension of the $U(N)$ principal chiral model with the Wess–Zumino term (nc-PCWZM). We extend the iterative procedure of Brezin *et al* [17] for the nc-PCWZM and derive a set of associated linear differential equations. It has been shown that the set of linear differential equations is equivalent to a noncommutative Lax formalism of the model. We derive a series of local and non-local conserved quantities. We expand the fields perturbatively and obtain zeroth- and first-order equations of motion, zero-curvature condition, Lax formalism, iterative procedure of Brezin *et al*, local and non-local conserved quantities.

The action for the principal chiral model with Wess–Zumino term is [14]

$$\frac{\beta}{2}\int \mathrm{d}^2x\,\mathrm{Tr}(\partial_+ g^{-1}\partial_- g) + \frac{\beta}{3}\kappa\int \mathrm{d}^2x\,\mathrm{Tr}(g^{-1}\,\mathrm{d}g)^3,$$

with the constraint

$$gg^{-1} = g^{-1}g = 1.$$

This is referred to as the principal chiral Wess–Zumino model (PCWZM). The constants β and κ are dimensionless and the field $g(x^{\pm})$ is a function of the space-time coordinate $x^{\pm}$. We define conserved currents of PCWZM associated with the global symmetry of the model as

$$\bar{j}_{\pm} \equiv (1\pm\kappa)j_{\pm} = -(1\pm\kappa)g^{-1}\,\partial_{\pm}g.$$

The equation of motion and zero-curvature condition for the PCWZM are

$$\partial_-\bar{j}_+ + \partial_+\bar{j}_- = 0, \qquad \partial_-\bar{j}_+ - \partial_+\bar{j}_- + [\bar{j}_+, \bar{j}_-] = 0,$$

and the above two equations give

$$\partial_-\bar{j}_+ = -\tfrac{1}{2}[\bar{j}_+, \bar{j}_-] = -\partial_+\bar{j}_-,$$

where $[\bar{j}_+, \bar{j}_-] = \bar{j}_+\bar{j}_- - \bar{j}_-\bar{j}_+$ is a commutator. The action for the principal chiral model with the Wess–Zumino term in the noncommutative space is obtained by replacing ordinary

[3] The product of two functions in a noncommutative space is defined as

$$(f\star g)(x) = f(x)g(x) + \frac{\mathrm{i}\theta^{ij}}{2}\partial_i f(x)\partial_j g(x) + \vartheta(\theta^2).$$

The following properties hold in a noncommutative space:

$$(f\star g)\star h = f\star(g\star h), \qquad f\star I = f = I\star f.$$

product of the fields with $\star$-product, i.e.

$$\frac{\beta}{2}\int \mathrm{d}^2x\, \mathrm{Tr}(\partial_+ g^{-1}\star\partial_- g) + \frac{\beta}{3}\kappa\int \mathrm{d}^2x\, \mathrm{Tr}(g^{-1}\star \mathrm{d}g)^3_\star,$$

with the constraint

$$g\star g^{-1} = g^{-1}\star g = 1.$$

This is referred to as the noncommutative principal chiral Wess–Zumino model (nc-PCWZM). The field $g(x^\pm)$ is a function of the space-time coordinate $x^\pm$ in the noncommutative space. The conserved currents for the nc-PCWZM are

$$\bar{j}^\star_\pm = (1\pm\kappa)j^\star_\pm = -(1\pm\kappa)g^{-1}\star\partial_\pm g.$$

The current conservation equation is

$$\partial_-\bar{j}^\star_+ + \partial_+\bar{j}^\star_- = 0, \tag{1}$$

and the zero-curvature condition is

$$\partial_-\bar{j}^\star_+ - \partial_+\bar{j}^\star_- + [\bar{j}^\star_+, \bar{j}^\star_-]_\star = 0, \tag{2}$$

where $[\bar{j}^\star_+, \bar{j}^\star_-]_\star = \bar{j}^\star_+\star\bar{j}^\star_- - \bar{j}^\star_-\star\bar{j}^\star_+$ is the commutator in the noncommutative space. The noncommutative equations (1) and (2) can be combined to give

$$\partial_-\bar{j}^\star_+ = -\tfrac{1}{2}[\bar{j}^\star_+, \bar{j}^\star_-]_\star = -\partial_+\bar{j}^\star_-. \tag{3}$$

It is straightforward to derive an infinite set of local conserved quantities from equation (3). We get a series of local conserved quantities

$$\partial_\mp \mathrm{Tr}(\bar{j}^\star_\pm)^n_\star = 0, \tag{4}$$

where the numbers n are precisely the exponents of the Lie algebra of $U(N)$. These conserved quantities are local functions of fields. The term local in this context is used for those conserved currents which depend on fields and their derivatives but not on their integrals. This should not be confused with the intrinsic non-locality of a noncommutative field theory emerging due to derivatives of fields to an infinite order, multiplied with the deformation parameter θ. This intrinsic non-locality due to noncommutativity persists in all definitions of conserved currents. In the ordinary commutative case, these conserved currents can be used to construct quantities which are in involution with each other and their spins are exponents of the Lie algebra modulo the coxter number [16]. It is, however, not straightforward to see in the noncommutative case, whether the conserved quantities obtained from equation (4) are in involution or not. The involution of the local conserved quantities can only be established if we know the Poisson bracket current algebra of nc-PCWZM. The canonical formalism is quite involved and does not allow us simply to write the noncommutative Poisson current algebra of the model.

In the nc-PCWZM, we also encounter with an infinite number of another type of conserved currents which are non-local as they depend non-locally on fields; i.e. they depend on fields, their derivatives and their integrals as well. In order to derive the non-local conserved currents, we define a covariant derivative in noncommutative space-time acting on some scalar field χ, such that

$$D_\pm\chi = (1\pm\kappa)(\partial_\pm\chi - j^\star_\pm\star\chi) \quad\Rightarrow\quad [D_+, D_-]_\star = 0, \tag{5}$$

where $[D_+, D_-]_\star = D_+ \star D_- - D_- \star D_+$. Now we suppose that there exist currents $\bar{j}_\pm^{\star(k)}$ for $k = 1, 2, \ldots, n$ which are conserved,

$$\partial_- \bar{j}_+^{\star(k)} + \partial_+ \bar{j}_-^{\star(k)} = 0, \tag{6}$$

such that

$$\bar{j}_\pm^{\star(k)} = \pm\partial_\pm \chi^{(k)}. \tag{7}$$

Further we have defined the $(k+1)$th current as

$$\begin{aligned}\bar{j}_\pm^{\star(k+1)} &= D_\pm \chi^{(k)} \\ &= (1 \pm \kappa)\partial_\pm \chi^{(k)} - \bar{j}_\pm^{\star} \star \chi^{(k)}.\end{aligned} \tag{8}$$

The current $\bar{j}_\pm^{\star(k+1)}$ is also conserved which can be checked as follows:

$$\begin{aligned}\partial_- \bar{j}_+^{\star(k+1)} + \partial_+ \bar{j}_-^{\star(k+1)} &= (\partial_- \star D_+ + \partial_+ \star D_-)\chi^{(k)} \\ &= (D_+ \star \partial_- + D_- \star \partial_+)\chi^{(k)} \\ &= -D_+ \star \bar{j}_-^{\star(k)} + D_- \star \bar{j}_+^{\star(k)} \\ &= -[D_+, D_-]_\star \chi^{(k-1)} \\ &= 0,\end{aligned}$$

where $D_\pm \star \partial_\pm = \partial_\pm \star D_\pm$. We set $\bar{j}_\pm^{\star(1)} = \bar{j}_\pm^{\star}$, $\bar{j}_\pm^{\star(0)} = 0$ and $\chi^{(0)} = 1$. Note that the conservation of the kth current implies the conservation of the $(k+1)$ th current and as a result an infinite number of conserved currents are obtained through induction.

The non-local conserved quantities in noncommutative space are now defined as

$$\bar{Q}^{\star(k)} = \int_{-\infty}^{\infty} \mathrm{d}y\, \bar{j}_0^{\star(k)}(t, y). \tag{9}$$

From equations (7) and (8), we get

$$\begin{aligned}\bar{j}_0^{\star(k+1)} &\equiv \tfrac{1}{2}\big(\bar{j}_+^{\star(k+1)} + \bar{j}_-^{\star(k+1)}\big) = \partial_0 \chi^{(k)} + \kappa \partial_1 \chi^{(k)} - \bar{j}_0^{\star} \star \chi^{(k)}, \\ \bar{j}_1^{\star(k+1)} &\equiv \tfrac{1}{2}\big(\bar{j}_+^{\star(k+1)} - \bar{j}_-^{\star(k+1)}\big) = \partial_1 \chi^{(k)} + \kappa \partial_0 \chi^{(k)} - \bar{j}_1^{\star} \star \chi^{(k)},\end{aligned}$$

where

$$\chi^{(k)}(t, y) = \int_{-\infty}^{y} \mathrm{d}z\, \bar{j}_0^{\star(k)}(t, z).$$

The second current is a non-local current; it depends non-locally on fields,

$$\bar{j}_0^{\star(2)}(t, y) = \bar{j}_1^{\star}(t, y) + \kappa \bar{j}_0^{\star}(t, y) - \bar{j}_0^{\star}(t, y) \star \int_{-\infty}^{y} \mathrm{d}z\, \bar{j}_0^{\star}(t, z). \tag{10}$$

The first two conserved charges are

$$\begin{aligned}\bar{Q}^{\star(1)} &= \int \mathrm{d}y\, \bar{j}_0^{\star}(t, y), \\ \bar{Q}^{\star(2)} &= \int \mathrm{d}y \left(\bar{j}_1^{\star}(t, y) + \kappa \bar{j}_0^{\star}(t, y) - \bar{j}_0^{\star}(t, y) \star \int_{-\infty}^{y} \mathrm{d}z\, \bar{j}_0^{\star}(t, z)\right).\end{aligned} \tag{11}$$

These conserved quantities reduce to the conserved quantities obtained in [7] in the limit when the deformation parameter approaches zero. In the limit when $\kappa \to 0$ the conserved quantities (11) reduce to the conserved quantities obtained for the noncommutative principal chiral model (nc-PCM) in [26].

We now relate the iterative procedure outlined above to the noncommutative Lax formalism of the nc-PCWZM. From equations (7) and (8), we have

$$\partial_\pm \chi^{(k)} = \pm D_\pm \chi^{(k-1)}.$$

Multiplying by λ^k and summing from $k = 1$ to $k = \infty$, we obtain

$$\sum_{k=1}^{\infty}(\lambda^k \partial_\pm \chi^{(k)}) = \pm \sum_{k=1}^{\infty}(\lambda^k D_\pm \chi^{(k-1)}).$$

As $\chi^{(0)} = 1$, the summation on the right-hand side can be extended to $k = 0$. The associated linear system for the nc-PCWZM can be written as

$$\partial_\pm u(t, x; \lambda) = \bar{A}_\pm^{\star(\lambda)} \star u(t, x; \lambda), \tag{12}$$

where the noncommutative fields $\bar{A}_\pm^{\star(\lambda)}$ are

$$\bar{A}_+^{\star(\lambda)} = \frac{-\lambda}{1 - \lambda(1 + \kappa)} \bar{j}_+^\star, \qquad \bar{A}_-^\star = \frac{\lambda}{1 + \lambda(1 - \kappa)} \bar{j}_-^\star,$$

and $u(t, x; \lambda)$ is expanded as

$$u(t, x; \lambda) = \sum_{k=0}^{\infty} \lambda^k \chi^{(k)}.$$

The compatibility condition of the linear system for the nc-PCWZM is the $\star$-zero-curvature condition

$$\left[\partial_+ - \bar{A}_+^{\star(\lambda)}, \partial_- - \bar{A}_-^{\star(\lambda)}\right]_\star \equiv \partial_- \bar{A}_+^{\star(\lambda)} - \partial_+ \bar{A}_-^{\star(\lambda)} + \left[\bar{A}_+^{\star(\lambda)}, \bar{A}_-^{\star(\lambda)}\right]_\star = 0. \tag{13}$$

Now we define Lax operators for the nc-PCWZM,

$$\bar{L}_\pm^{\star(\lambda)} = \partial_\pm - \bar{A}_\pm^{\star(\lambda)}, \tag{14}$$

obeying the following equations:

$$\partial_\mp \bar{L}_\pm^{\star(\lambda)} = \left[\bar{A}_\mp^{\star(\lambda)}, \bar{L}_\pm^{\star(\lambda)}\right]_\star.$$

The associated linear system (12) can be re-expressed in terms of space-time coordinates as

$$\partial_0 u(t, x; \lambda) = \bar{A}_0^{\star(\lambda)} \star u(t, x; \lambda), \qquad \partial_1 u(t, x; \lambda) = \bar{A}_1^{\star(\lambda)} \star u(t, x; \lambda), \tag{15}$$

with the noncommutative fields $\bar{A}_0^{\star(\lambda)}$ and $\bar{A}_1^{\star(\lambda)}$ given by

$$\bar{A}_0^{\star(\lambda)} = \tfrac{1}{2}\left(\bar{A}_+^{\star(\lambda)} + \bar{A}_-^{\star(\lambda)}\right), \qquad \bar{A}_1^{\star(\lambda)} = \tfrac{1}{2}\left(\bar{A}_+^{\star(\lambda)} - \bar{A}_-^{\star(\lambda)}\right).$$

The associated Lax operator $\bar{L}_0^{\star(\lambda)} = \partial_0 - \bar{A}_0^{\star(\lambda)}$ obeys the following Lax equation

$$\partial_0 \bar{L}_1^{\star(\lambda)} = \left[\bar{A}_0^{\star(\lambda)}, \bar{L}_1^{\star(\lambda)}\right]_\star.$$

This gives the time involution of the Lax operator related to the isospectral problem of the model. We note that the Lax formalism can be generalized to its noncommutative version without any constraints.

Let $\hat{g}$ be a solution of equation of motion; therefore, the associated linear system for the fields $\hat{g}$ can be written as

$$\partial_\pm \hat{u}(t, x; \lambda) = \hat{\bar{A}}_\pm^{\star(\lambda)} \star \hat{u}(t, x; \lambda),$$

where the noncommutative fields $\hat{\bar{A}}_\pm^{\star(\lambda)}$ are given by

$$\hat{\bar{A}}_+^{\star(\lambda)} = \frac{-\lambda}{1 - \lambda(1 + \kappa)} \hat{\bar{j}}_+^\star, \qquad \hat{\bar{A}}_-^\star = \frac{\lambda}{1 + \lambda(1 - \kappa)} \hat{\bar{j}}_-^\star.$$

The compatibility condition of the above linear system is

$$\left[\partial_+ - \hat{\bar{A}}_+^{\star(\lambda)}, \partial_- - \hat{\bar{A}}_-^{\star(\lambda)}\right]_\star \equiv \partial_- \hat{\bar{A}}_+^{\star(\lambda)} - \partial_+ \hat{\bar{A}}_-^{\star(\lambda)} + \left[\hat{\bar{A}}_+^{\star(\lambda)}, \hat{\bar{A}}_-^{\star(\lambda)}\right]_\star = 0. \tag{16}$$

The Darboux transformation relates the matrix function $\hat{u}(t, x; \lambda)$ and $u(t, x; \lambda)$ by

$$\hat{u} = u \star v, \tag{17}$$

and one can easily obtain

$$\partial_{\pm} v = \hat{\bar{A}}_{\pm}^{\star(\lambda)} \star v - v \star \bar{A}_{\pm}^{\star(\lambda)}. \tag{18}$$

Equation (18) is also known as the Bäcklund transformation.

An infinite number of non-local conserved quantities can also be generated from the Lax formalism of nc-PCWZM. We assume the spatial boundary conditions such that the currents $\bar{j}_{\pm}^{\star}$ vanish as $x \to \pm\infty$. Equation (15) implies that $u(t, \infty; \lambda)$ are time independent. The residual freedom in the solution for $u(t, \infty; \lambda)$ allows us to fix $u(t, \infty; \lambda) = 1$ the unit matrix, and then we are left with the time-independent matrix-valued function

$$\bar{Q}^{\star}(\lambda) = u(t, \infty; \lambda). \tag{19}$$

Expanding $\bar{Q}^{\star}(\lambda)$ as power series in λ gives an infinite number of non-local conserved quantities

$$\bar{Q}^{\star}(\lambda) = \sum_{k=o}^{\infty} \lambda^k \bar{Q}^{\star(k)}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \bar{Q}^{\star(k)} = 0. \tag{20}$$

For the explicit expressions of the non-local conserved quantities, we write (15) as

$$u(t, x; \lambda) = 1 - \frac{1}{2} \int_{-\infty}^{x} \mathrm{d}y \big(\bar{A}_{+}^{\star(\lambda)}(t, y) - \bar{A}_{-}^{\star(\lambda)}(t, y)\big) \star u(t, y; \lambda). \tag{21}$$

When we expand the field $u(t, x; \lambda)$ as power series in λ as

$$u(t, x; \lambda) = \sum_{k=o}^{\infty} \lambda^k u_k(t, x), \tag{22}$$

and compare the coefficients of powers of λ, we get a series of conserved non-local currents, which upon integration give non-local conserved quantities (11). In the commutative case, the local conserved quantities based on the invariant tensors, all Poisson commute with each other and with the non-local conserved quantities. The classical Poisson brackets of non-local conserved quantities constitute classical Yangian symmetry [16]. In the noncommutative case we expect similar results but once we know the Poisson bracket algebra of the currents $\bar{j}_{\pm}^{\star}$, we shall be able to address the Poisson bracket algebra of these conserved quantities.

We now study the perturbative expansion of the field $g(x^{\pm})$ and compute the equation of motion and zero-curvature condition, Lax equation and the conserved quantities up to first order in the deformation parameter θ. We define the group-valued field $g(x^{\pm})$ by [38]

$$g = \exp_{\star}\left(\frac{\mathrm{i}\varphi}{2}\right), \qquad \text{and} \qquad g^{-1} = \exp_{\star}\left(-\frac{\mathrm{i}\varphi}{2}\right),$$

with

$$g = \exp_{\star}\left(\frac{\mathrm{i}\varphi}{2}\right) = 1 + \frac{\mathrm{i}}{1!}\left(\frac{\varphi}{2}\right) + \frac{1}{2!}\left(\frac{\mathrm{i}}{2}\right)^2 \varphi \star \varphi + \cdots,$$

where φ is in the Lie algebra of $U(N)$. The components of Noether's currents are

$$\bar{j}_{\pm}^{\star} = -(1 \pm \kappa) g^{-1} \star \partial_{\pm} g,$$

or

$$\bar{j}_{\pm}^{\star} = -(1 \pm \kappa)\frac{\mathrm{i}}{2}\partial_{\pm}\varphi - (1 \pm \kappa)^2 \frac{\theta}{2!}\left(\frac{\mathrm{i}}{2}\right)^2 \left(\partial_{\pm}^2 \varphi \partial_{\mp}\varphi - \partial_{\mp}\partial_{\pm}\varphi\partial_{\pm}\varphi\right) + \vartheta(\theta^2). \tag{23}$$

We expand φ as a power series in θ

$$\varphi = \varphi^{[0]} + \theta\varphi^{[1]}, \qquad \bar{j}^{\star}_{\pm} = \bar{j}^{[0]}_{\pm} + \theta\tilde{\bar{j}}^{[1]}_{\pm}, \tag{24}$$

where

$$\bar{j}^{[0]}_{\pm} = -\frac{\mathrm{i}}{2}(1\pm\kappa)\partial_{\pm}\varphi^{[0]}, \qquad \bar{j}^{[1]}_{\pm} = -\frac{\mathrm{i}}{2}(1\pm\kappa)\partial_{\pm}\varphi^{[1]},$$
$$\tilde{\bar{j}}^{[1]}_{\pm} = \bar{j}^{[1]}_{\pm} - \frac{1}{2!}\left(\bar{j}^{[0]}_{\pm\pm}\bar{j}^{[0]}_{\mp} - \frac{1}{2}\bar{j}^{[0]}_{\mp}\bar{j}^{[0]^2}_{\pm} + \frac{1}{2}\bar{j}^{[0]}_{\pm}\bar{j}^{[0]}_{\mp}\bar{j}^{[0]}_{\pm}\right).$$

By substituting the value of $\bar{j}^{\star}_{\pm}$ from equation (24) into equations of motion (1) and (2), we obtain

$$\partial_-\bar{j}^{[0]}_+ + \partial_+\bar{j}^{[0]}_- = 0,$$
$$\partial_-\tilde{\bar{j}}^{[1]}_+ + \partial_+\tilde{\bar{j}}^{[1]}_- = 0,$$
$$\partial_-\bar{j}^{[0]}_+ - \partial_+\bar{j}^{[0]}_- + \left[\bar{j}^{[0]}_+, \bar{j}^{[0]}_-\right] = 0,$$
$$\partial_-\tilde{\bar{j}}^{[1]}_+ - \partial_+\tilde{\bar{j}}^{[1]}_- + \left[\tilde{\bar{j}}^{[1]}_+, \bar{j}^{[0]}_-\right] + \left[\bar{j}^{[0]}_+, \tilde{\bar{j}}^{[1]}_-\right] = -\frac{\mathrm{i}}{2}\left(\bar{j}^{[0]}_{++}\bar{j}^{[0]}_{--} + \bar{j}^{[0]}_{--}\bar{j}^{[0]}_{++}\right) - \frac{\mathrm{i}}{8}\left[\bar{j}^{[0]}_+, \bar{j}^{[0]}_-\right]^2.$$

It is clear from the above equations that currents are conserved up to first order in θ but the zero-curvature condition is spoiled. The perturbative expansion of the iterative construction is

$$\bar{j}^{[0](k+1)}_{\pm} = D^{[0]}_{\pm}\chi^{[0](k)} \quad \Rightarrow \quad \partial_-\bar{j}^{[0](k+1)}_+ + \partial_+\bar{j}^{[0](k+1)}_- = 0,$$
$$\tilde{\bar{j}}^{[1](k+1)}_{\pm} = D^{[0]}_{\pm}\chi^{[1](k)} - \tilde{D}^{[1]}_{\pm}\chi^{[0](k)} \quad \Rightarrow \quad \partial_-\tilde{\bar{j}}^{[1](k+1)}_+ + \partial_+\tilde{\bar{j}}^{[1](k+1)}_- = 0,$$

where

$$D^{[0]}_{\pm}\chi^{[1](k)} = \partial_{\pm}\chi^{[1](k)} - \bar{j}^{[0]}_{\pm}\chi^{[1](k)},$$
$$\tilde{D}^{[1]}_{\pm}\chi^{[0](k)} = \tilde{\bar{j}}^{[1]}_{\pm}\chi^{[0](k)} + \frac{\mathrm{i}}{2}\left(\partial_{\pm}\bar{j}^{[0]}_{\pm}\partial_{\mp} - \partial_{\mp}\bar{j}^{[0]}_{\pm}\partial_{\pm}\right)\chi^{[0](k)}.$$

From this analysis, it is obvious that the conservation of the kth current implies the conservation of the $(k+1)$th current up to first order in θ.

The zero-curvature condition (13) for the fields $\bar{A}^{\star(\lambda)}_{\pm}$ in the perturbative expansion becomes

$$\partial_-\bar{A}^{[0](\lambda)}_+ - \partial_+\bar{A}^{[0](\lambda)}_- + \left[\bar{A}^{[0](\lambda)}_+, \bar{A}^{[0](\lambda)}_-\right] = 0, \tag{25}$$

$$\begin{aligned}\partial_-\tilde{\bar{A}}^{[1](\lambda)}_+ - \partial_+\tilde{\bar{A}}^{[1](\lambda)}_- + \left[\tilde{\bar{A}}^{[1](\lambda)}_+, \bar{A}^{[0](\lambda)}_-\right] + \left[\bar{A}^{[0]}_+, \tilde{\bar{A}}^{[1](\lambda)}_-\right] &= -\frac{\mathrm{i}}{2}\left(\bar{A}^{[0](\lambda)}_{++}\bar{A}^{[0](\lambda)}_{--} + \bar{A}^{[0](\lambda)}_{--}\bar{A}^{[0](\lambda)}_{++}\right)\\ &\quad + \frac{\mathrm{i}}{2}\left(\partial_-\bar{A}^{[0](\lambda)}_+\partial_+\bar{A}^{[0](\lambda)}_- + \partial_+\bar{A}^{[0](\lambda)}_-\partial_-\bar{A}^{[0](\lambda)}_+\right),\end{aligned} \tag{26}$$

where the fields $\bar{A}^{[0](\lambda)}_{\pm}$ and $\tilde{\bar{A}}^{[1](\lambda)}_{\pm}$ are given by

$$\bar{A}^{[0](\lambda)}_+ = \frac{-\lambda}{1-\lambda(1+\kappa)}\bar{j}^{[0]}_+, \qquad \bar{A}^{[0]}_- = \frac{\lambda}{1+\lambda(1-\kappa)}\bar{j}^{[0]}_-,$$
$$\tilde{\bar{A}}^{[1](\lambda)}_+ = \frac{-\lambda}{1-\lambda(1+\kappa)}\tilde{\bar{j}}^{[1]}_+, \qquad \tilde{\bar{A}}^{[1]}_- = \frac{\lambda}{1+\lambda(1-\kappa)}\tilde{\bar{j}}^{[1]}_-.$$

The perturbative expansion of the linear system (12) is

$$\partial_{\pm}u^{[0]}(x,t;\lambda) = \bar{A}^{[0](\lambda)}_{\pm}u^{[0]}(x,t;\lambda),$$
$$\begin{aligned}\partial_{\pm}u^{[1]}(x,t;\lambda) &= \bar{A}^{[0](\lambda)}_{\pm}u^{[1]}(x,t;\lambda)\\ &\quad + \left(\tilde{\bar{A}}^{[1](\lambda)}_{\pm} + \frac{\mathrm{i}}{2}\left(\bar{A}^{[0](\lambda)}_{\pm\pm}\bar{A}^{[0](\lambda)}_{\mp} - \bar{A}^{[0](\lambda)}_{\pm\mp}\bar{A}^{[0](\lambda)}_{\pm}\right)\right)u^{[0]}(x,t;\lambda).\end{aligned}$$

The compatibility conditions of first and second equations in the above set of equations are (25) and (26), respectively. The perturbative expansion of the Lax operators (14) is expressed as

$$\bar{L}_{\pm}^{(\lambda)} = \bar{L}_{\pm}^{[0](\lambda)} + \theta \tilde{\bar{L}}_{\pm}^{[1](\lambda)},$$

where $\bar{L}_{\pm}^{[0](\lambda)}$ and $\tilde{\bar{L}}_{\pm}^{[1](\lambda)}$ are given by

$$\bar{L}_{\pm}^{[0](\lambda)} = \partial_{\pm} - \bar{A}_{\pm}^{[0](\lambda)}, \qquad \tilde{\bar{L}}_{\pm}^{[1](\lambda)} = -\tilde{\bar{A}}_{\pm}^{[1](\lambda)},$$

obeying the following equations:

$$\partial_{\mp} \bar{L}_{\pm}^{[0](\lambda)} = \left[\bar{A}_{\mp}^{[0](\lambda)}, \bar{L}_{\pm}^{[0](\lambda)}\right], \tag{27}$$

$$\begin{aligned}\partial_{\mp} \tilde{\bar{L}}_{\pm}^{[1](\lambda)} &= \left[\bar{A}_{\mp}^{[0](\lambda)}, \tilde{\bar{L}}_{\pm}^{[1](\lambda)}\right] + \left[\tilde{\bar{A}}_{\mp}^{[1](\lambda)}, \bar{L}_{\pm}^{[0](\lambda)}\right] - \frac{\mathrm{i}}{2}\left(\bar{L}_{\pm\pm}^{[0](\lambda)} \bar{A}_{\mp\mp}^{[0](\lambda)} + \bar{A}_{\mp\mp}^{[0](\lambda)} \bar{L}_{\pm\pm}^{[0](\lambda)}\right) \\ &\quad + \frac{\mathrm{i}}{2}\left(\partial_{\mp} \bar{L}_{\pm}^{[0](\lambda)} \partial_{\pm} \bar{A}_{\mp}^{[0](\lambda)} + \partial_{\pm} \bar{A}_{\mp}^{[0](\lambda)} \partial_{\mp} \bar{L}_{\pm}^{[0](\lambda)}\right).\end{aligned} \tag{28}$$

The Lax equations (27) and (28) are equivalent to (25) and (26), respectively.

By substituting the value of $\bar{j}_{\pm}^{\star}$ from equation (24) into equation (4), we find the following zeroth-order and first-order local conservation laws,

$$\begin{aligned}
&\partial_{\mp} \operatorname{Tr}\left(\bar{j}_{\pm}^{[0]2}\right) = 0, \\
&\partial_{\mp} \operatorname{Tr}\left(\bar{j}_{\pm}^{[0]} \tilde{\bar{j}}_{\pm}^{[1]}\right) = 0, \\
&\partial_{\mp} \operatorname{Tr}\left(\bar{j}_{\pm}^{[0]3}\right) = 0, \\
&\partial_{\mp} \operatorname{Tr}\left(\bar{j}_{\pm}^{[0]2} \tilde{\bar{j}}_{\pm}^{[1]} - \tfrac{1}{4} \bar{j}_{\pm}^{[0]} \bar{j}_{\pm\pm}^{[0]}\left[\bar{j}_{\pm}^{[0]}, \bar{j}_{\mp}^{[0]}\right] + \tfrac{1}{4} \bar{j}_{\pm\pm}^{[0]} \bar{j}_{\pm}^{[0]}\left[\bar{j}_{\pm}^{[0]}, \bar{j}_{\mp}^{[0]}\right]\right) = 0, \\
&\partial_{\mp} \operatorname{Tr}\left(\bar{j}_{\pm}^{[0]4}\right) = 0, \\
&\partial_{\mp} \operatorname{Tr}\left(\bar{j}_{\pm}^{[0]3} \tilde{\bar{j}}_{\pm}^{[1]} - \tfrac{1}{8} \bar{j}_{\pm}^{[0]2} \bar{j}_{\pm\pm}^{[0]}\left[\bar{j}_{\pm}^{[0]}, \bar{j}_{\mp}^{[0]}\right] + \tfrac{1}{8} \bar{j}_{\pm\pm}^{[0]} \bar{j}_{\pm}^{[0]2}\left[\bar{j}_{\pm}^{[0]}, \bar{j}_{\mp}^{[0]}\right]\right) = 0.
\end{aligned}$$

Similarly, we can expand the iterative procedure perturbatively up to first order in the deformation parameter θ. The first two non-local conserved currents are

$$\begin{aligned}
\bar{j}_0^{(1)[0]}(t, y) &= \bar{j}_0^{[0]}(t, y), \\
\tilde{\bar{j}}_0^{(1)[1]}(t, y) &= \tilde{\bar{j}}_0^{[1]}(t, y), \\
\bar{j}_0^{(2)[0]}(t, y) &= \bar{j}_1^{[0]}(t, y) + \kappa \bar{j}_0^{[0]}(t, y) - \bar{j}_0^{[0]}(t, y) \int_{-\infty}^{y} \bar{j}_0^{[0]}(t, z)\, \mathrm{d}z, \\
\tilde{\bar{j}}_0^{(2)[1]}(t, y) &= \tilde{\bar{j}}_1^{[1]}(t, y) + \kappa \tilde{\bar{j}}_0^{[1]}(t, y) - \tilde{\bar{j}}_1^{[1]}(t, y) \int_{-\infty}^{y} \bar{j}_0^{[0]}(t, z)\, \mathrm{d}z \\
&\quad - \bar{j}_1^{[0]}(t, y) \int_{-\infty}^{y} \tilde{\bar{j}}_0^{[1]}(t, z)\, \mathrm{d}z + \text{total derivative},
\end{aligned}$$

and the corresponding conserved quantities are

$$\begin{aligned}
\bar{Q}^{(1)[0]} &= \int_{-\infty}^{\infty} \bar{j}_0^{[0]}(t, y)\, \mathrm{d}y, \\
\tilde{\bar{Q}}^{(1)[1]} &= \int_{-\infty}^{\infty} \tilde{\bar{j}}_0^{[1]}(t, y)\, \mathrm{d}y, \\
\bar{Q}^{(2)[0]} &= \int_{-\infty}^{\infty} \left(\bar{j}_1^{[0]}(t, y) + \kappa \bar{j}_0^{[0]}(t, y) - \bar{j}_0^{[0]}(t, y) \int_{-\infty}^{y} \bar{j}_0^{[0]}(t, z)\, \mathrm{d}z\right) \mathrm{d}y,
\end{aligned}$$

$$\tilde{\bar{Q}}^{(2)[1]} = \int_{-\infty}^{\infty} \left(\tilde{\bar{j}}_1^{[1]}(t, y) + \kappa \tilde{\bar{j}}_0^{[1]}(t, y) - \tilde{\bar{j}}_1^{[1]}(t, y) \int_{-\infty}^{y} \bar{j}_0^{[0]}(t, z)\, \mathrm{d}z - \bar{j}_1^{[0]}(t, y) \int_{-\infty}^{y} \tilde{\bar{j}}_0^{[1]}(t, z)\, \mathrm{d}z \right) \mathrm{d}y.$$

Note that the first-order correction to the first non-local conserved quantity is an integral of non-local function of the fields.

In summary, we have extended the $U(N)$ principal chiral model with the Wess–Zumino term in noncommutative space and have shown that the nc-PCWZM preserves its integrability without any extra constraint. A set of nontrivial local and non-local conserved quantities are calculated for the nc-PCWZM. We check the integrability of the model at a perturbative level and write the zeroth- and first-order equations of motion, zero-curvature condition, local and non-local conserved quantities. Similar noncommutative extension can also be investigated for the supersymmetric principal chiral model and the WZW model. Another interesting direction to pursue is to investigate $\star$-Poisson bracket algebra of the currents of nc-PCWZM and to check the involution of local conserved quantities.

Acknowledgments

The authors acknowledge the enabling role of the Higher Education Commission Islamabad, Pakistan and appreciate its financial support through ‘Merit Scholarship Scheme for PhD studies in Science and Technology (200 Scholarships)’.

References

[1] Babelon O, Bernand D and Talon M 2003 *Introduction to Classical Integrable Systems* (Cambridge: Cambridge University Press)
Abdalla E, Abdalla M C B and Rothe K 1991 *Non-Perturbative Methods in Two-Dimensional Quantum Field Theory* (Singapore: World Scientific)
Lamb G L Jr 1980 *Elements of Soliton Theory* (New York: Wiley)
Das A 1989 *Integrable Models* (Singapore: World Scientific)
Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[2] Pohlmeyer K 1976 *Commun. Math. Phys.* **46** 207
[3] Saleem U and Hassan M 2005 *Eur. Phys. J.* C **38** 521
[4] Siddiq M and Hassan M 2005 *Eur. Phys. Lett.* **70** 149
[5] Siddiq M and Hassan M 2005 *Chin. Phys. Lett.* **22** 1567
[6] Lüscher M and Pohlmeyer K 1978 *Nucl. Phys.* B **137** 46
[7] Abdalla E and Abdalla M C B 1985 *Phys. Lett.* B **152** 59
[8] de Vega H J 1979 *Phys. Lett.* B **87** 233
[9] Bernard D 1992 *Phys. Lett.* B **279** 78
[10] Bhattacharya G and Rajeev S 1984 *Nucl. Phys.* B **246** 157
[11] MacKay N J 1992 *Phys. Lett.* B **281** 90
[12] Levi D, Ragnisco O and Ragnisco O 1982 *Sym. Lett. Nuovo Cimento* **33** 401
[13] Levi D and Ragnisco O 1982 *Phys. Lett.* A **87** 381
[14] Witten E 1984 *Commun. Math. Phys.* **92** 455
[15] Ogielski A T 1980 *Phys. Rev.* D **21** 406
[16] Evans J M, Hassan M, MacKay N J and Mountain A J 1999 *Nucl. Phys.* B **561** 385
[17] Brézin E, Itzykson C, Zinn-Justin J and Zuber J-B 1979 *Phys. Lett.* B **82** 442
[18] Connes A 1994 *Noncommutative Geometry* (New York: Academic)
[19] Minwalla S, Raamsdonk M V and Seiberg N 2000 *J. High Energy Phys.* JHEP02(2000)020
[20] Seiberg N and Witten E 1999 *J. High Energy Phys.* JHEP09(1999)032
[21] Susskind L 2001 The quantum Hall fluid and noncommutative Cherm–Simons theorem *Preprint* hep-th/0101029
[22] Gomis J and Mehen T 2000 *Nucl. Phys.* B **591** 265

[23] Gomis J and Aharony O 2000 *J. High Energy Phys.* JHEP09(2000)023
[24] Seiberg N, Susskind L and Toumbas N 2000 *J. High Energy Phys.* JHEP06(2000)044
[25] Furuta K and Inami T 2000 *Mod. Phys. Lett.* A **15** 997
[26] Moriconi M and Carnero I C 2003 *Nucl. Phys.* B **673** 437
[27] Profumo S 2002 *J. High Energy Phys.* JHEP10(2002)035
[28] Grisaru M T and Penati S 2003 *Nucl. Phys.* B **655** 250
[29] Saleem U, Siddiq M and Hassan M 2005 *Chin. Phys. Lett.* **22** 1076
[30] Grisaru M T and Penati S 2004 *J. High Energy Phys.* JHEP04(2004)057
[31] Hamanaka M 2005 *J. Math. Phys.* **46** 052701
[32] Hamanaka M and Toda K 2003 *J. Phys. A: Math. Gen.* **36** 11981
[33] Hamanaka M and Toda K 2003 *Phys. Lett.* A **316** 77
[34] Dimakis A and Hoissen F M 2000 *Int. J. Mod. Phys.* B **14** 2455
[35] Dimakis A and Hoissen F M 2000 Noncommutative Korteweg–de-Vries equation *Preprint* hep-th/0007074
[36] Dimakis A and Hoissen F M 2004 *J. Phys. A: Math. Gen.* **37** 4069
[37] Dimakis A and Hoissen F M 2000 A noncommutative version of the nonlinear Schrödinger equation *Preprint* hep-th/0007015
[38] Jack I, Jones D R T and Mohammedi N 2001 *Phys. Lett.* B **520** 405
[39] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99